Communication-Avoiding Algorithms for Linear Algebra, ML and Beyond

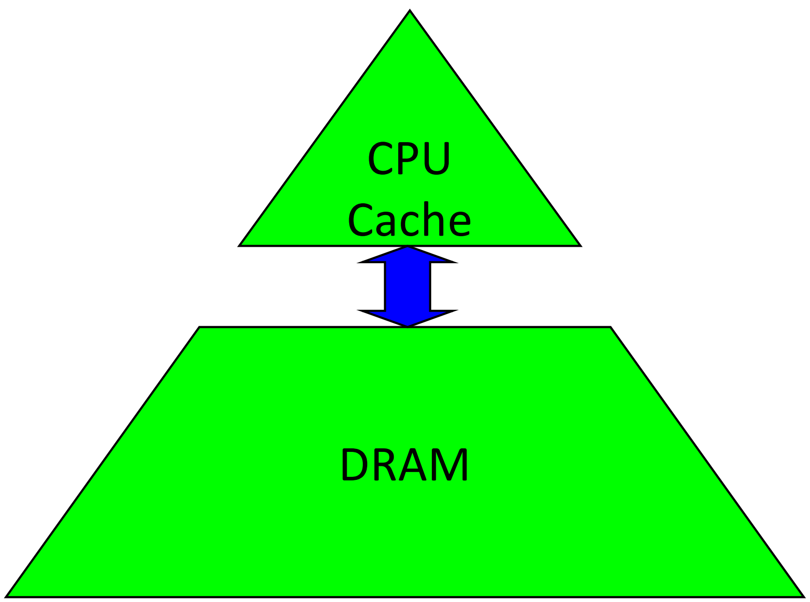
Jim Demmel, EECS & Math Depts., UC Berkeley

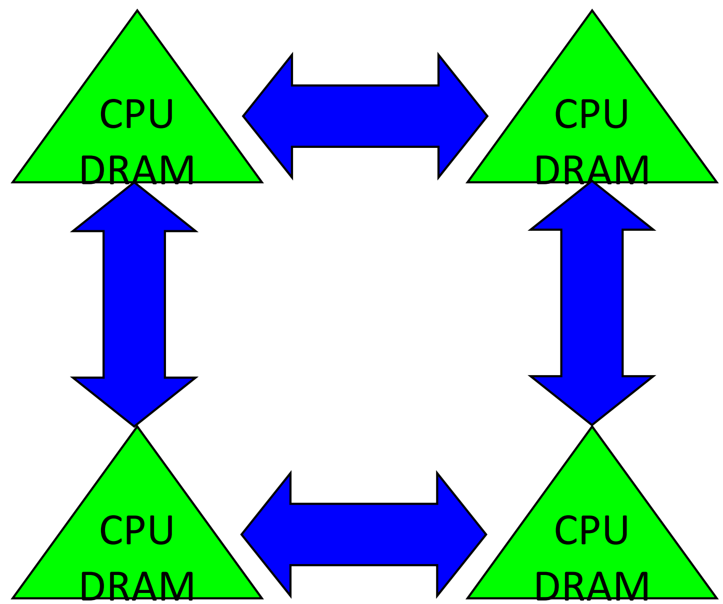
And many, many others …

Why avoid communication? (1/3)

Algorithms have two costs (measured in time or energy):

1. Arithmetic (FLOPS)
2. Communication: moving data between
   - levels of a memory hierarchy (sequential case)
   - processors over a network (parallel case).



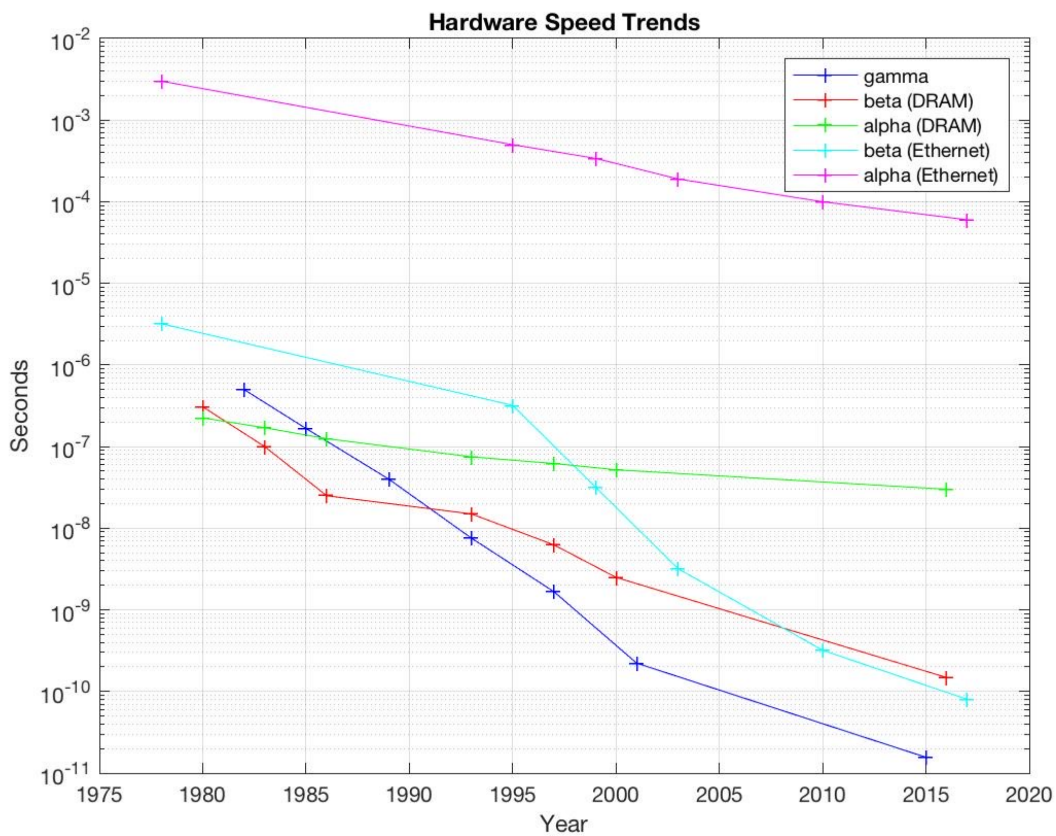


Why avoid communication? (2/3)

- Running time of an algorithm is sum of 3 terms:
  - # flops * time_per_flop
  - # words moved / bandwidth
  - # messages * latency

  (the last two terms: communication)

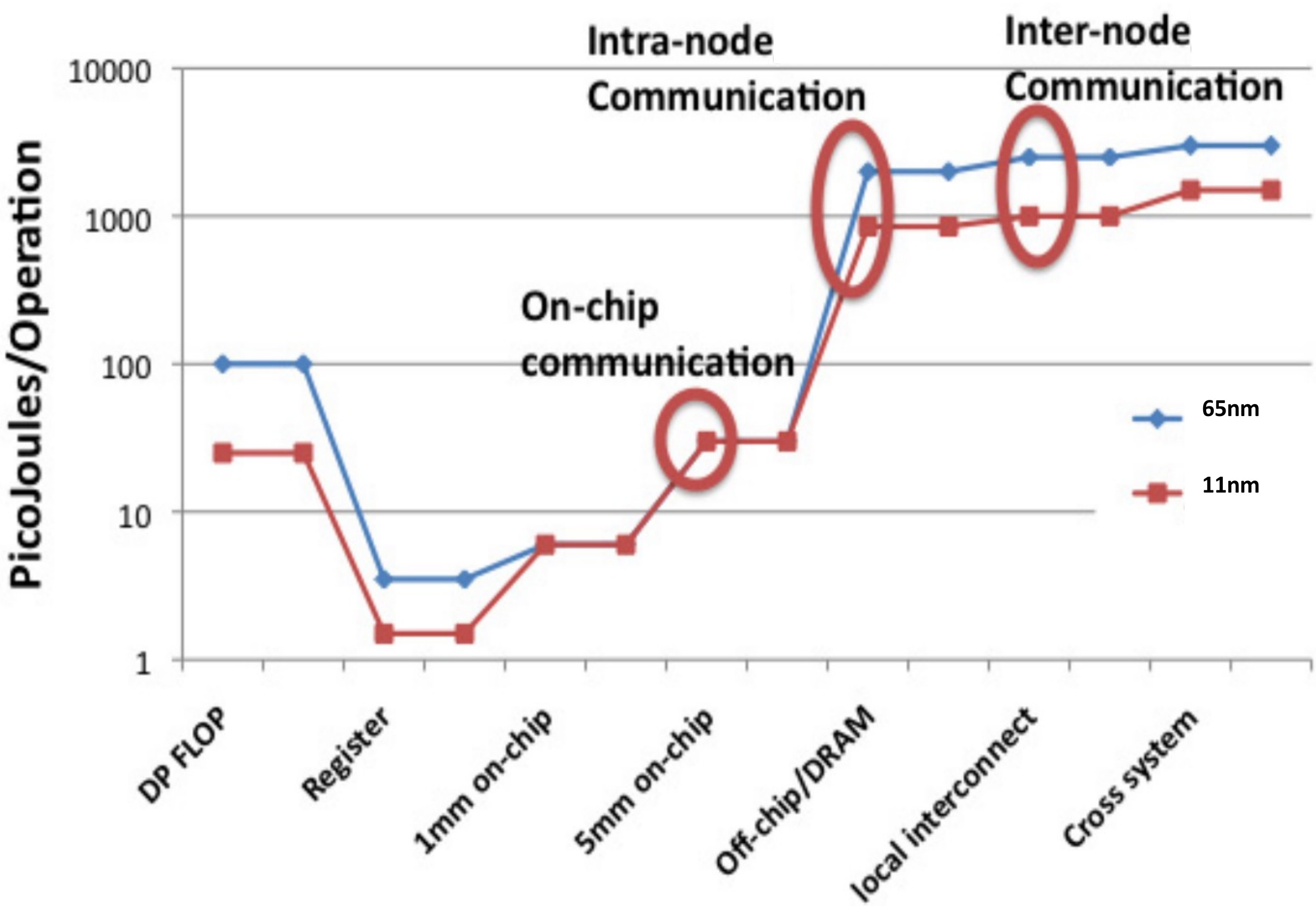
- Time_per_flop (γ) << 1/ bandwidth (β) << latency (α)



Data from Patterson & Hennessey, 2019

Why avoid communication? (3/3)

Same story for saving energy



Goals

- Redesign algorithms to *avoid* communication
  - Between all memory hierarchy levels
    - L1 $\longleftrightarrow$ L2 $\longleftrightarrow$ DRAM $\longleftrightarrow$ network, etc
- Attain lower bounds if possible
  - Classical algorithms often far from lower bounds
  - Large speedups and energy savings possible
- Automate implementation of communication-avoiding (CA) algorithms

Sample Speedups

- Doing same operations, just in a different order
  - Up to **12x** faster for 2.5D dense matmul on 64K core IBM BG/P
  - Up to **100x** faster for 1.5D sparse-dense matmul on 1536 core Cray XC30
  - Up to **6.2x** faster for 2.5D All-Pairs-Shortest-Path on 24K core Cray XE6
  - Up to **11.8x** faster for direct N-body on 32K core IBM BG/P
- Mathematically identical answer, but different algorithm
  - Up to **13x** faster for Tall Skinny QR on Tesla C2050 Fermi NVIDIA GPU
  - Up to **6.7x** faster for symeig(band A) on 10 core Intel Westmere
  - Up to **4.2x** faster for BiCGStab (MiniGMG bottom solver) on 24K core Cray XE6
  - Up to **5.1x** faster for coordinate descent LASSO on 3K core Cray XC30
- Different algorithm, different approximate answer
  - Up to **16x** faster for SVM on a 1536 core Cray XC30
  - Up to **135x** faster for ImageNet training on 2K Intel KNL nodes

Sample Speedups

- Doing same operations, just in a different order

Ideas adopted by Nervana, "deep learning" startup, acquired by Intel in August 2016

Kwasniewski, Hoefler, et al (Best Student Paper, SC'19)

- Mathematically identical answer, but different algorithm

SIAG on Supercomputing Best Paper Prize, 2016

(D., Grigori, Hoemmen, Langou)

Released in LAPACK 3.7, 2016

LAPACK 3.10: Householder Reconstruction, 2021

- Different algorithm, different approximate answer

IPDPS 2015 Best Paper Prize (You, D. Czechowski, Song, Vuduc)

ICPP 2018 Best Paper Prize (You, Zhang, Hsieh, D., Keutzer)

2019: Idea (LARS) adopted by industry standard benchmark MLPerf

Outline

- Linear Algebra
  - Communication Lower Bounds for classical direct linear algebra
  - CA 2.5D Matmul
  - TSQR - Tall-Skinny QR
  - Iterative Methods for linear algebra
- Machine Learning
  - Training Neural Nets – “ImageNet training in minutes”
  - Convolutional Neural Nets
- And Beyond
  - Extending communication lower bounds and optimal algorithms to general loop nests

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Summary of CA Linear Algebra

- "Direct" Linear Algebra
  - Lower bounds on communication for linear algebra problems like Ax=b, least squares, Ax = λx, SVD, etc
  - Only some attained by algorithms in standard libraries
    - LAPACK, ScaLAPACK, …
  - New algorithms needed to attain these lower bounds
    - New numerical properties, ways to encode answers, data structures, not just loop transformations
  - Autotuning to find optimal implementation (eg GPTune)
  - Sparse matrices: depends on sparsity structure
- Ditto for "Iterative" Linear Algebra

Lower bound for all "n^3-like" linear algebra

- Let M = "fast" memory size (per processor)

#words_moved (per processor) = Ω(#flops (per processor) / $M^{1/2}$)

- Parallel case: assume either load or memory balanced
- Holds for
  - Matmul

Lower bound for all “n^3-like” linear algebra

- Let M = “fast” memory size (per processor)

#words_moved (per processor) = Ω(#flops (per processor) / $M^{1/2}$)

#messages_sent ≥ #words_moved / largest_message_size

- Parallel case: assume either load or memory balanced

- Holds for
  - Matmul, BLAS, LU, QR, eig, SVD, tensor contractions, …
  - Some whole programs (sequences of these operations, no matter how individual ops are interleaved, eg A^k)
  - Dense and sparse matrices (where #flops << n^3)
  - Sequential and parallel algorithms
  - Some graph-theoretic algorithms (eg Floyd-Warshall)

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SIAM SIAG/Linear Algebra Prize, 2012

(Ballard, D., Holtz, Schwartz)

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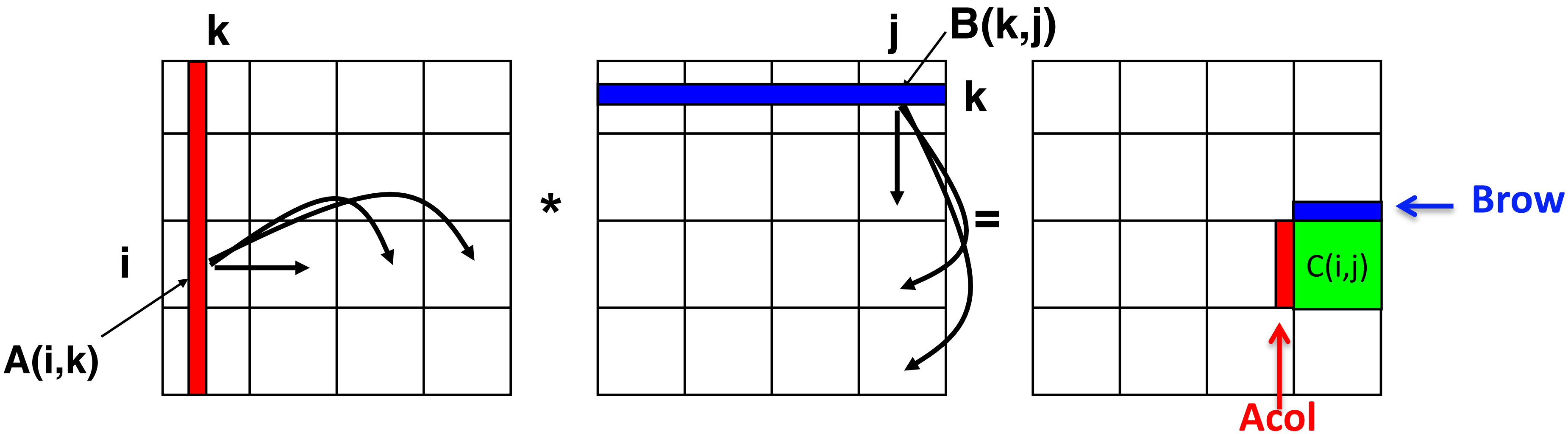
SUMMA– n x n matmul on $P^{1/2}$ x $P^{1/2}$ grid

(nearly) optimal using minimum memory $M=O(n^2/P)$



For k=0 to n/b-1 … b = block size = #cols in A(i,k) = #rows in B(k,j)

for all i = 1 to $P^{1/2}$

owner of A(i,k) broadcasts it to whole processor row (using binary tree)

for all j = 1 to $P^{1/2}$

owner of B(k,j) broadcasts it to whole processor column (using bin. tree)

Receive A(i,k) into Acol

Receive B(k,j) into Brow

C_myproc = C_myproc + Acol * Brow

Summary of dense *parallel* algorithms attaining communication lower bounds

- Assume nxn matrices on P processors
- Minimum Memory per processor = $M = O(n^2 / P)$
- Recall lower bounds:

  #words_moved = $\Omega((n^3/ P) / M^{1/2}) = \Omega(n^2 / P^{1/2})$

  #messages = $\Omega((n^3/ P) / M^{3/2}) = \Omega(P^{1/2})$
- SUMMA attains this lower bound
- Does ScaLAPACK attain these bounds?
  - For #words_moved: mostly, except nonsym. Eigenproblem
  - For #messages: asymptotically worse, except Cholesky
- New algorithms attain all bounds, up to polylog(P) factors
  - Cholesky, LU, QR, Sym. and Nonsym eigenproblems, SVD

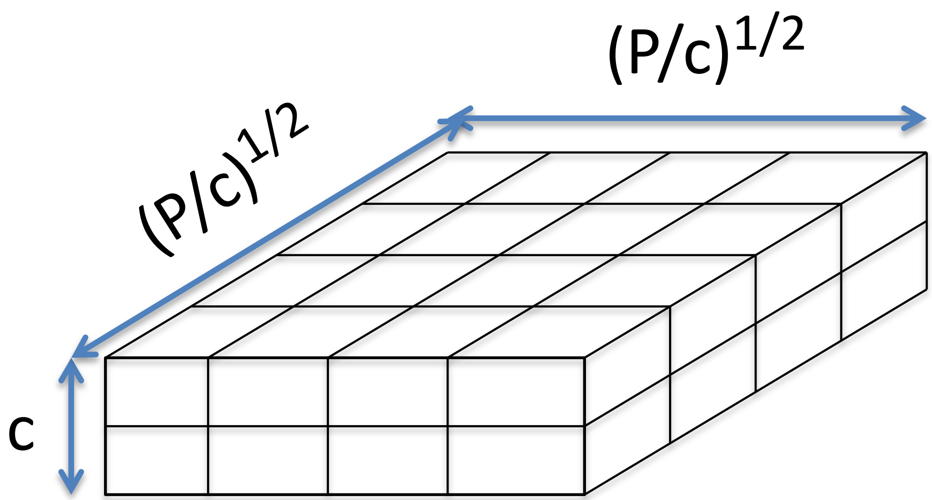
Can we do Better?

Can we do better?

- Aren't we already optimal?
- Why assume $M = O(n^2/p)$, i.e. minimal?
  - Lower bound still true if more memory
  - Can we attain it?
- Special case: "3D Matmul"
  - Uses $M = O(n^2/p^{2/3})$
  - Dekel, Nassimi, Sahni [81], Bernsten [89], Agarwal, Chandra, Snir [90], Johnson [93], Agarwal, Balle, Gustavson, Joshi, Palkar [95]
- Not always $p^{1/3}$ times as much memory available…

2.5D Matrix Multiplication

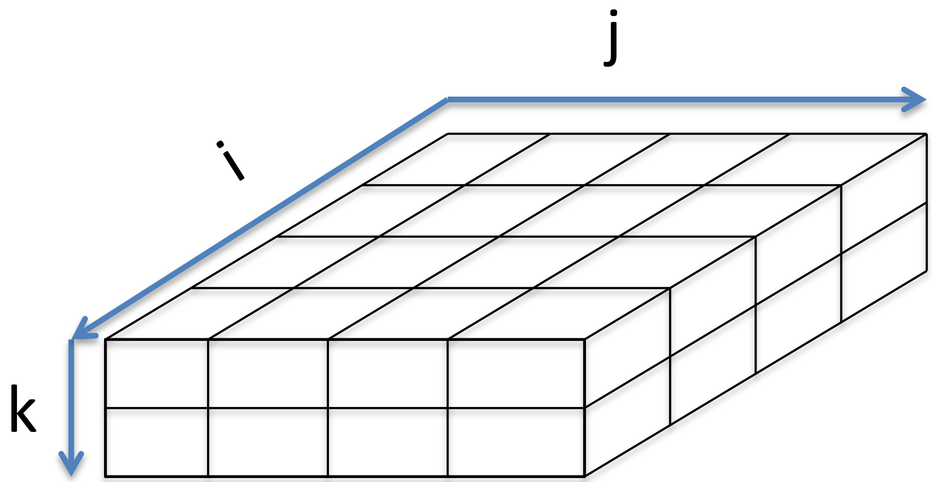
- Assume can fit cn^2/P data per processor, $c > 1$
- Processors form $(P/c)^{1/2}$ x $(P/c)^{1/2}$ x c grid



Example: P = 32, c = 2

2.5D Matrix Multiplication

- Assume can fit cn^2/P data per processor, $c > 1$
- Processors form $(P/c)^{1/2}$ x $(P/c)^{1/2}$ x c grid

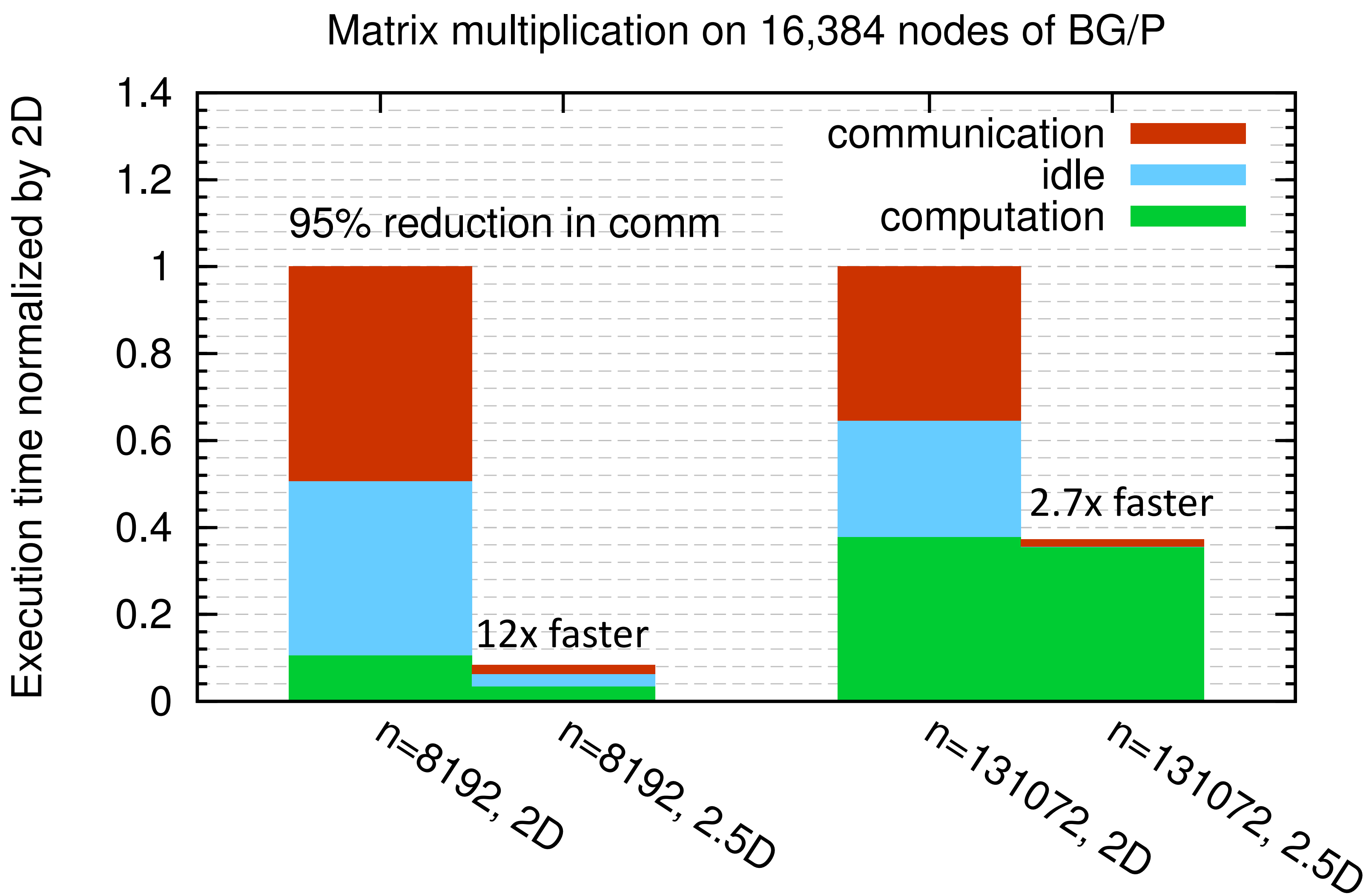


Initially P(i,j,0) owns A(i,j) and B(i,j) each of size $n(c/P)^{1/2}$ x $n(c/P)^{1/2}$

(1) P(i,j,0) broadcasts A(i,j) and B(i,j) to P(i,j,k)
(2) Processors at level k perform 1/c-th of SUMMA, i.e. 1/c-th of Σ_m A(i,m)*B(m,j)
(3) Sum-reduce partial sums Σ_m A(i,m)*B(m,j) along k-axis so P(i,j,0) owns C(i,j)

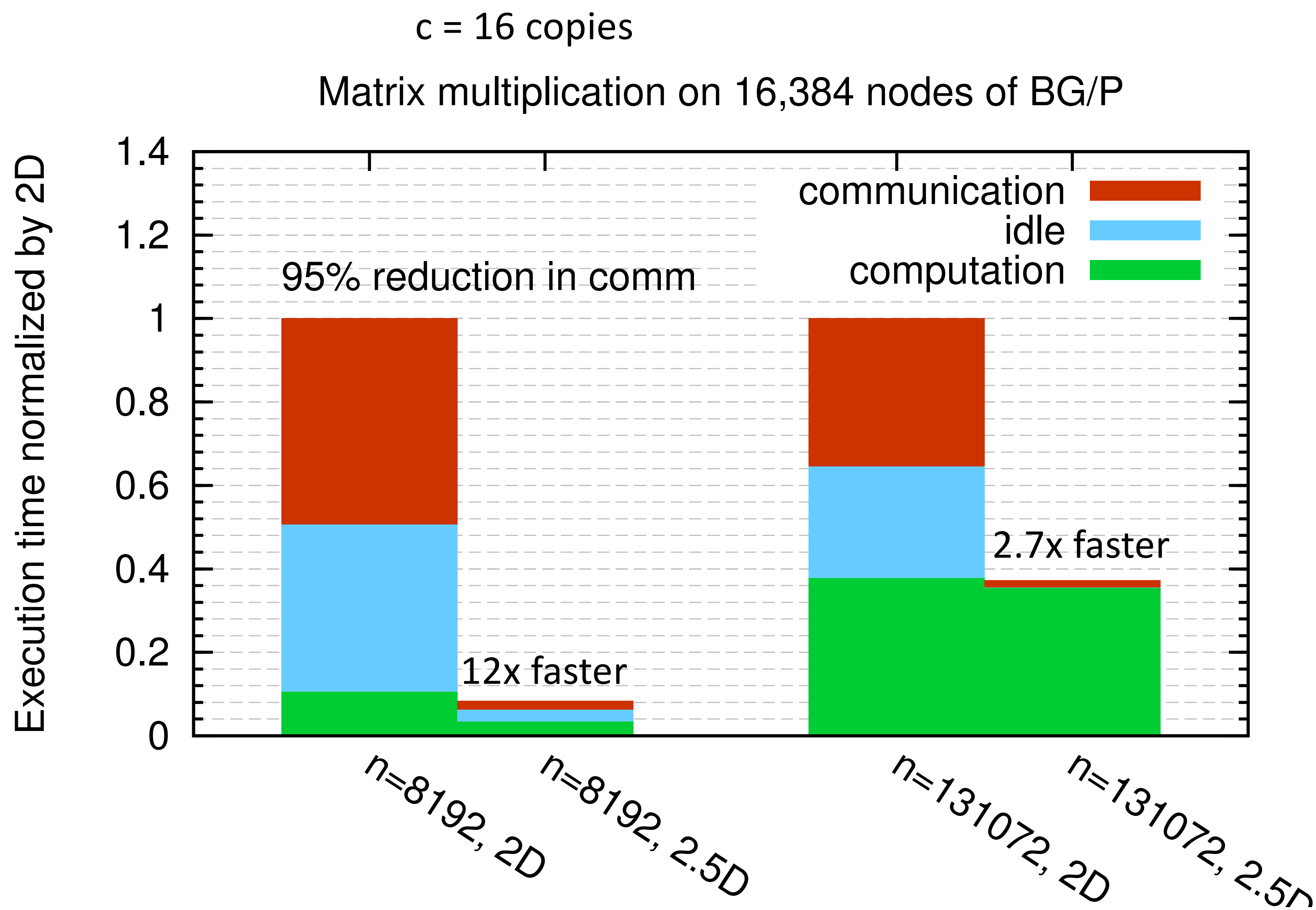
2.5D Matmul on BG/P, 16K nodes / 64K cores

c = 16 copies



Matmul achieves perfect strong scaling in time and energy

2.5D Matmul on BG/P, 16K nodes / 64K cores



Distinguished Paper Award, EuroPar'11 (Solomonik, D.)

Kwasniewski, Hoefler, et al (Best Student Paper, SC'19)

Beyond 2.5D Matmul

- Same idea extends to LU, QR, Jacobi, other algs
  - With a catch: can reduce bandwidth, but not latency
  - Bandwidth * Latency = $\Omega(n^2)$
- N-body problem
  - for i=1:n, f(i)=0, for j=1:n, f(i) = f(i) + force(p(i),p(j))
  - Bandwidth and latency both decrease with extra memory, attainable lower bounds
  - Same idea applies to Transformers
    - arxiv.org/pdf/2407.00611

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TSQR: QR of a Tall, Skinny matrix

$$W = \begin{pmatrix} W_0 \\ \hline W_1 \\ \hline W_2 \\ \hline W_3 \end{pmatrix}$$

$$\begin{pmatrix} R_{00} \\ R_{10} \\ \hline R_{20} \\ R_{30} \end{pmatrix} = \begin{pmatrix} Q_{01}\, R_{01} \\ \hline Q_{11}\, R_{11} \end{pmatrix}$$

$$\begin{pmatrix} R_{01} \\ \hline R_{11} \end{pmatrix} = \begin{pmatrix} Q_{02}\, R_{02} \end{pmatrix}$$

TSQR: QR of a Tall, Skinny matrix

$$W = \begin{pmatrix} W_0 \\ \hline W_1 \\ \hline W_2 \\ \hline W_3 \end{pmatrix} = \begin{pmatrix} Q_{00} R_{00} \\ \hline Q_{10} R_{10} \\ \hline Q_{20} R_{20} \\ \hline Q_{30} R_{30} \end{pmatrix} = \begin{pmatrix} Q_{00} & & & \\ \hline & Q_{10} & & \\ \hline & & Q_{20} & \\ \hline & & & Q_{30} \end{pmatrix} \cdot \begin{pmatrix} R_{00} \\ \hline R_{10} \\ \hline R_{20} \\ \hline R_{30} \end{pmatrix}$$

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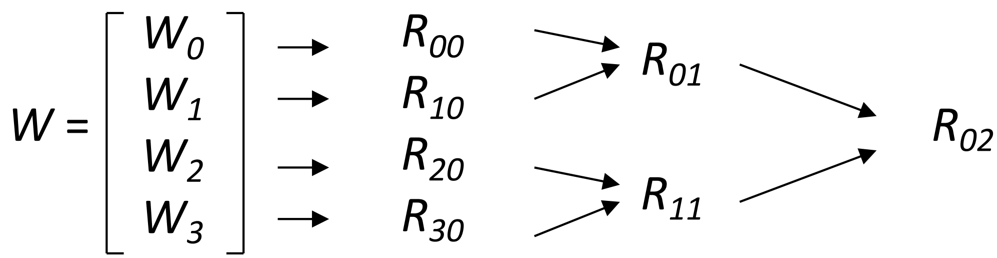
$$\begin{pmatrix} R_{01} \\ \hline R_{11} \end{pmatrix} = \begin{pmatrix} Q_{02} R_{02} \end{pmatrix}$$

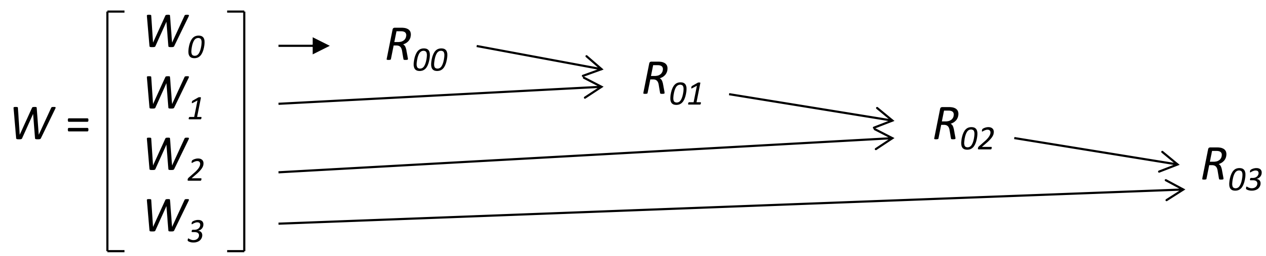
Output = $\{ Q_{00}, Q_{10}, Q_{20}, Q_{30}, Q_{01}, Q_{11}, Q_{02}, R_{02} \}$

TSQR: An Architecture-Dependent Algorithm

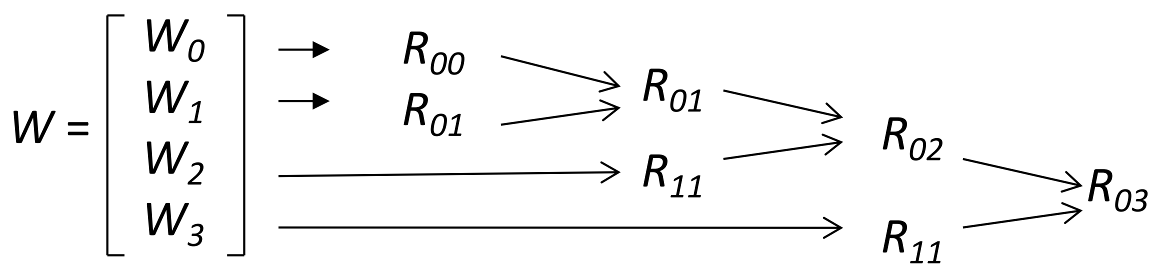
Parallel:



Sequential:



Dual Core:



Multicore / Multisocket / Multirack / Multisite / Out-of-core: ?

Can choose reduction tree dynamically

TSQR Performance Results

- Parallel
  - Intel Clovertown
    - Up to **8x** speedup (8 core, dual socket, 10M x 10)
  - Pentium III cluster, Dolphin Interconnect, MPICH
    - Up to **6.7x** speedup (16 procs, 100K x 200)
  - BlueGene/L
    - Up to **4x** speedup (32 procs, 1M x 50)
  - Tesla C 2050 / Fermi
    - Up to **13x** (110,592 x 100)
  - Grid – **4x** on 4 cities vs 1 city (Dongarra, Langou et al)
  - Cloud – **1.6x slower than just accessing data twice** (Gleich and Benson)
- Sequential
  - "**Infinite speedup**" for out-of-core on PowerPC laptop
    - As little as 2x slowdown vs (predicted) infinite DRAM
    - LAPACK with virtual memory never finished
- SVD costs about the same
- Joint work with Grigori, Hoemmen, Langou, Anderson, Ballard, Keutzer, others

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Avoiding Communication in Iterative Linear Algebra

- k-steps of iterative solver for sparse Ax=b or Ax=λx
  - Does k SpMVs with A and starting vector
  - Many such "Krylov Subspace Methods"
    - Conjugate Gradients (CG), GMRES, Lanczos, Arnoldi, …
- Goal: minimize communication
  - Assume matrix "well-partitioned"
  - Serial implementation
    - Conventional: O(k) moves of data from slow to fast memory
    - **New**: **O(1) moves of data – optimal**
  - Parallel implementation on p processors
    - Conventional: O(k log p) messages (k SpMV calls, dot prods)
    - **New: O(log p) messages - optimal**
- Lots of speed up possible (modeled and measured)
  - Price: some redundant computation
  - Challenges: Poor partitioning, Preconditioning, Num. Stability

Minimizing Communication of GMRES to solve Ax=b

- GMRES: find x in span{b,Ab,…,A^kb} minimizing || Ax-b ||$_2$

Standard GMRES

```
for i=1 to k
  w = A · v(i-1)   … SpMV
  MGS(w, v(0),…,v(i-1))
  update v(i), H
endfor
solve LSQ problem with H
```

Communication-avoiding GMRES

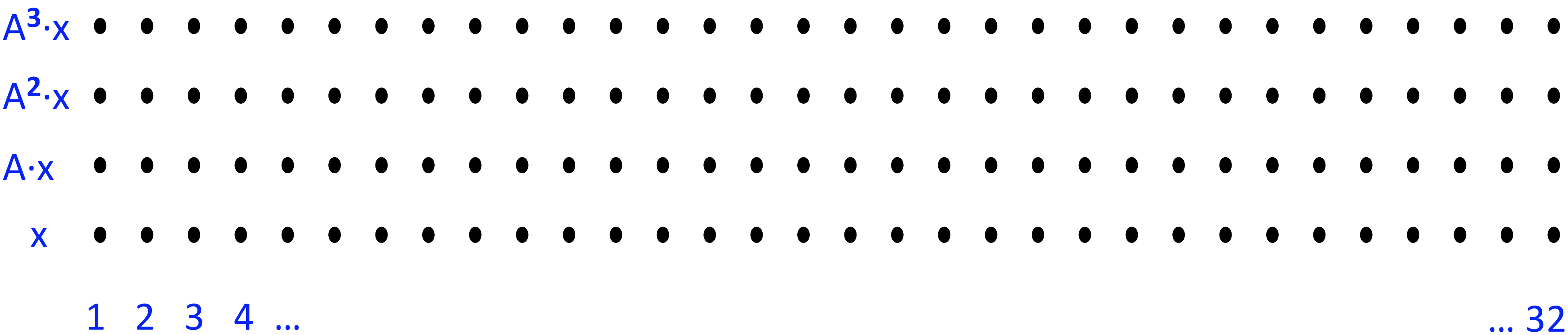
```
W = [ v, Av, A^2v, … , A^kv ]
[Q,R] = TSQR(W)
     … "Tall Skinny QR"
build H from R
solve LSQ problem with H
```

Sequential case: #words moved decreases by a factor of k
Parallel case: #messages decreases by a factor of k

- Oops – W from power method, precision lost!
- Fix: replace W by [v, p_1(A)v, p_2(A)v, …, p_k(A)v]
- Up to **2.3x** speedup for GMRES on 8 core Intel Clovertown (Hoemmen)
- Up to **4.2x** speedup for BiCGStab on 24K core Cray XE6 (Carson)

Communication Avoiding Kernels:
The Matrix Powers Kernel : $[Ax, A^2x, \ldots, A^kx]$

- Replace k iterations of $y = A \cdot x$ with $[Ax, A^2x, \ldots, A^kx]$



- Example: A tridiagonal, n=32, k=3
- Works for any "well-partitioned" A

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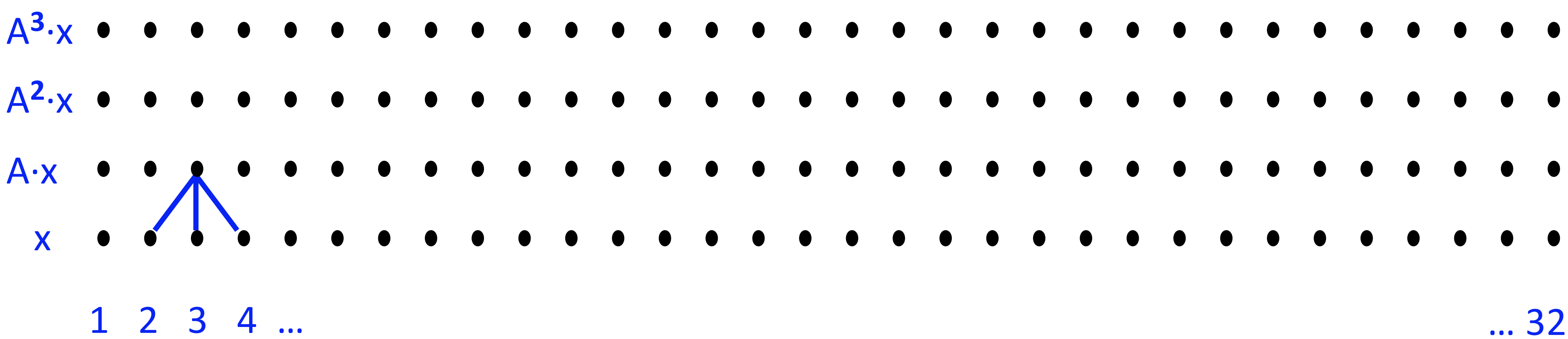

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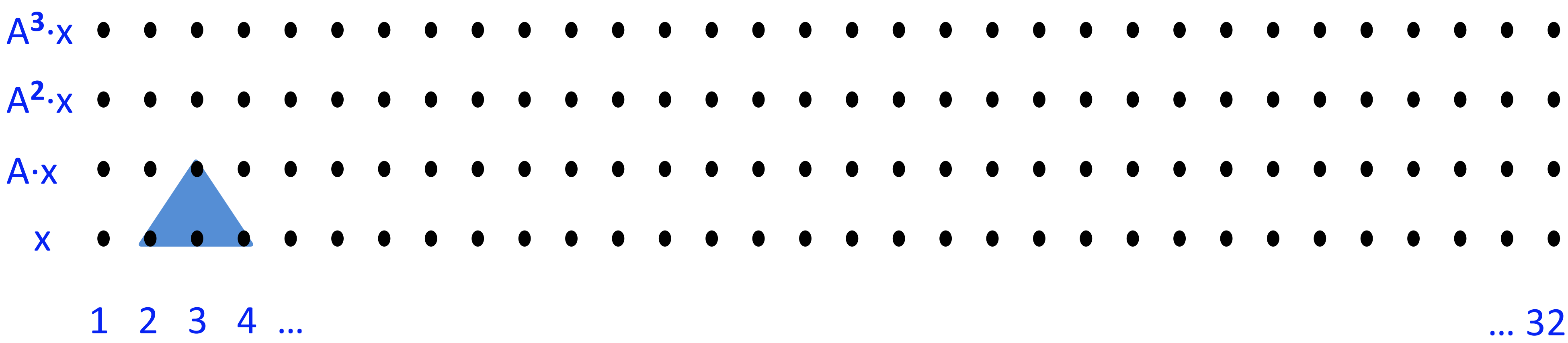

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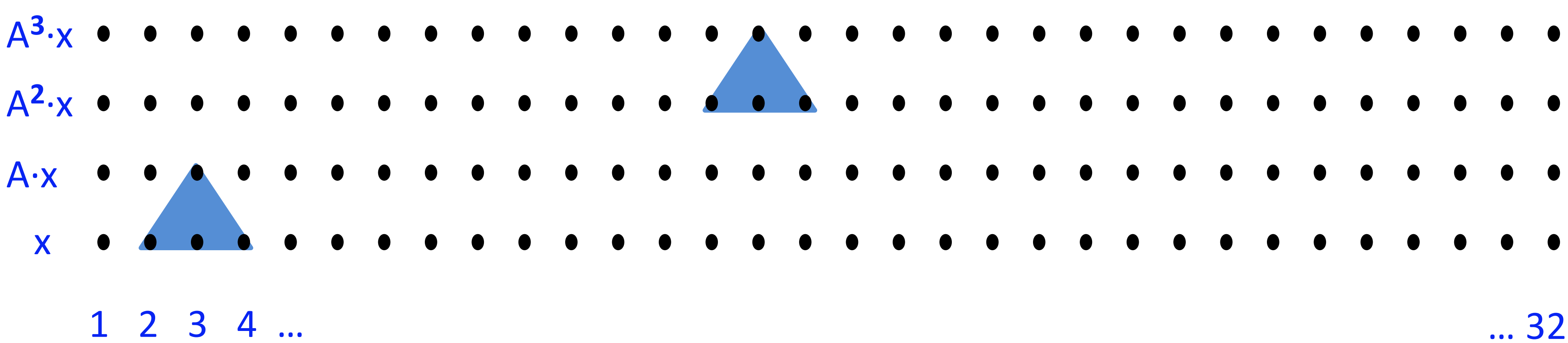

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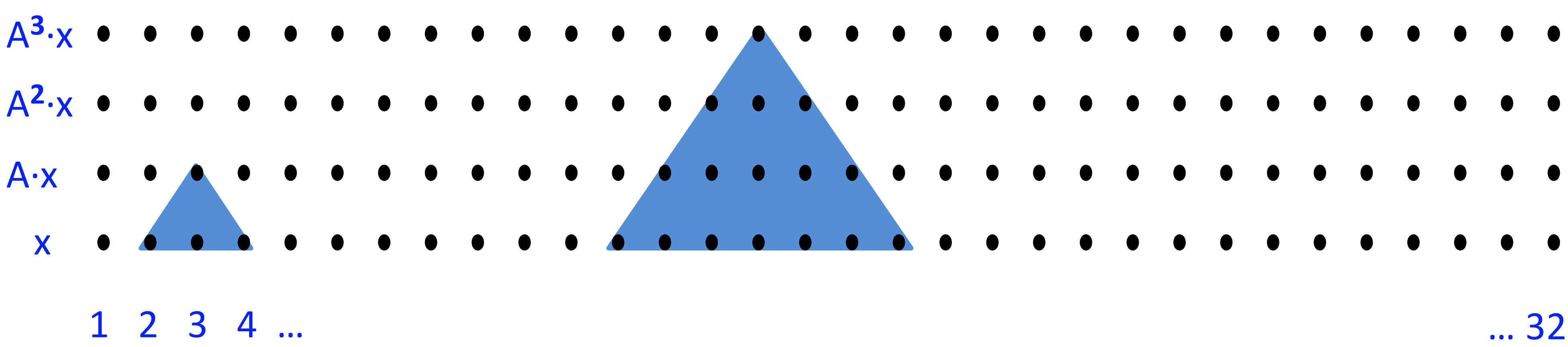

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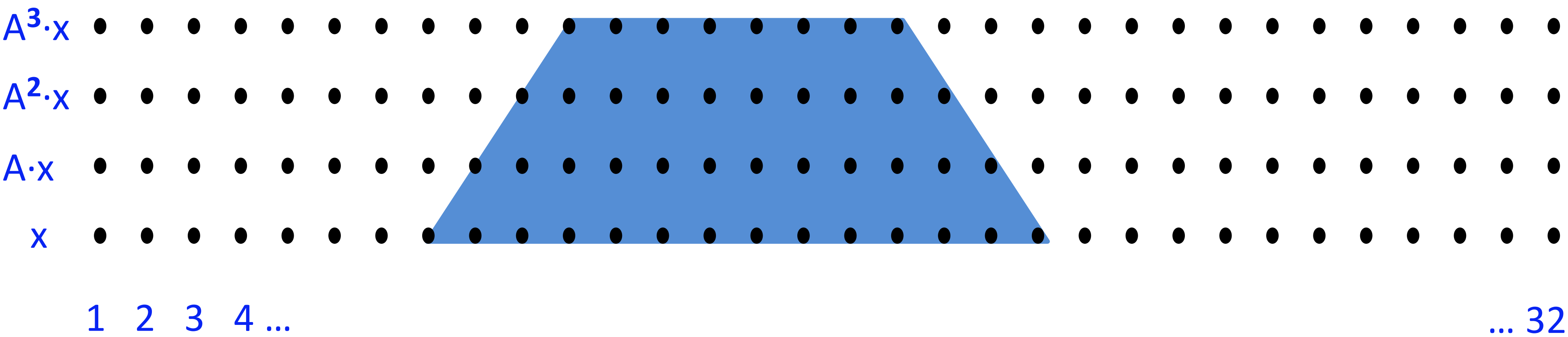

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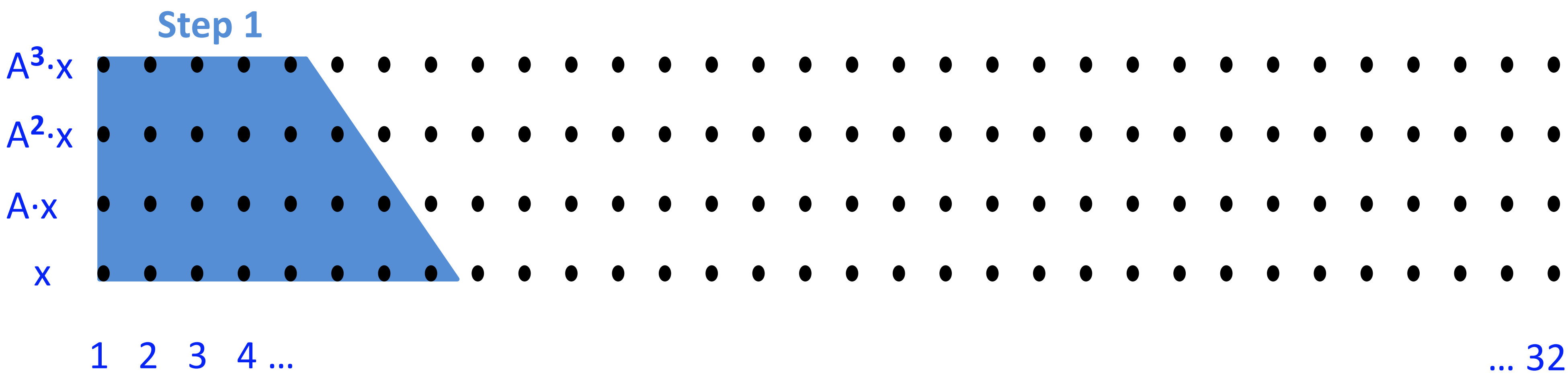
- Sequential Algorithm



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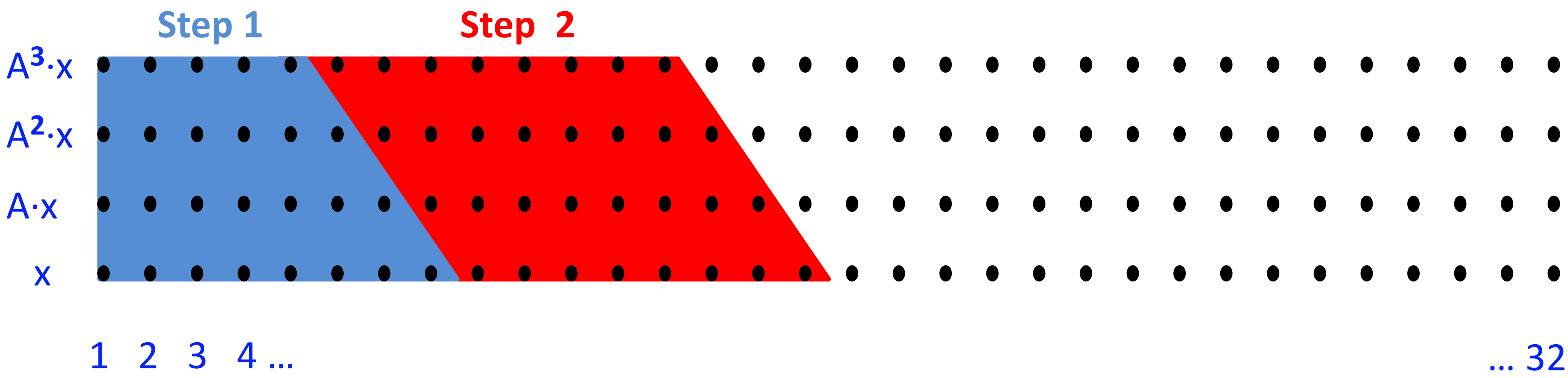

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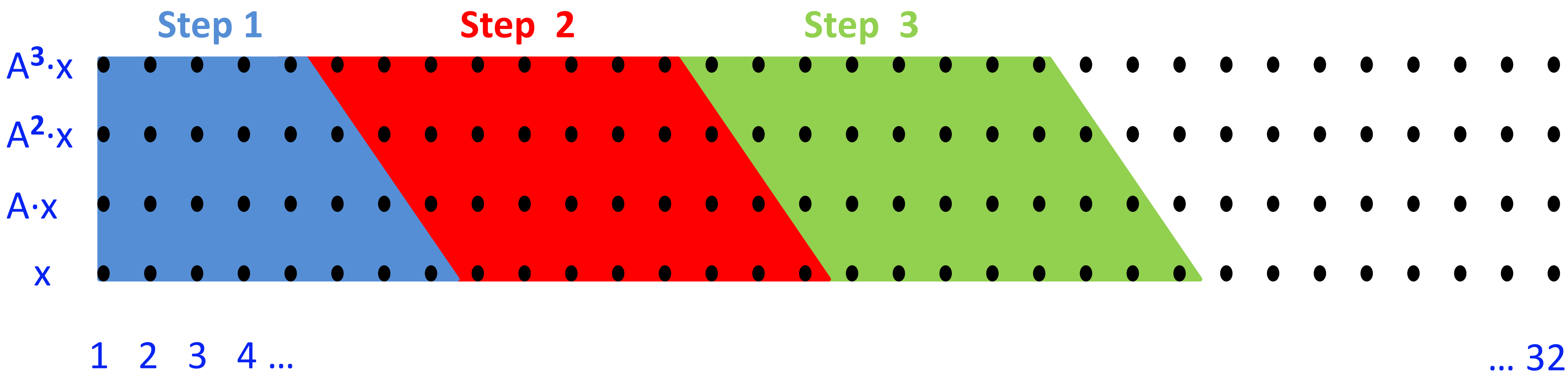

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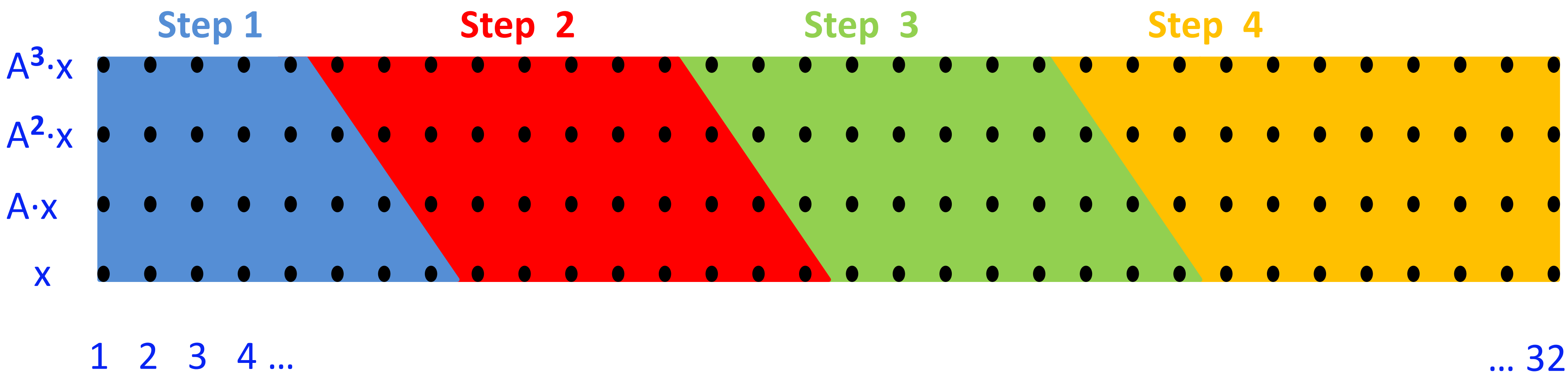

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Communication Avoiding Kernels:

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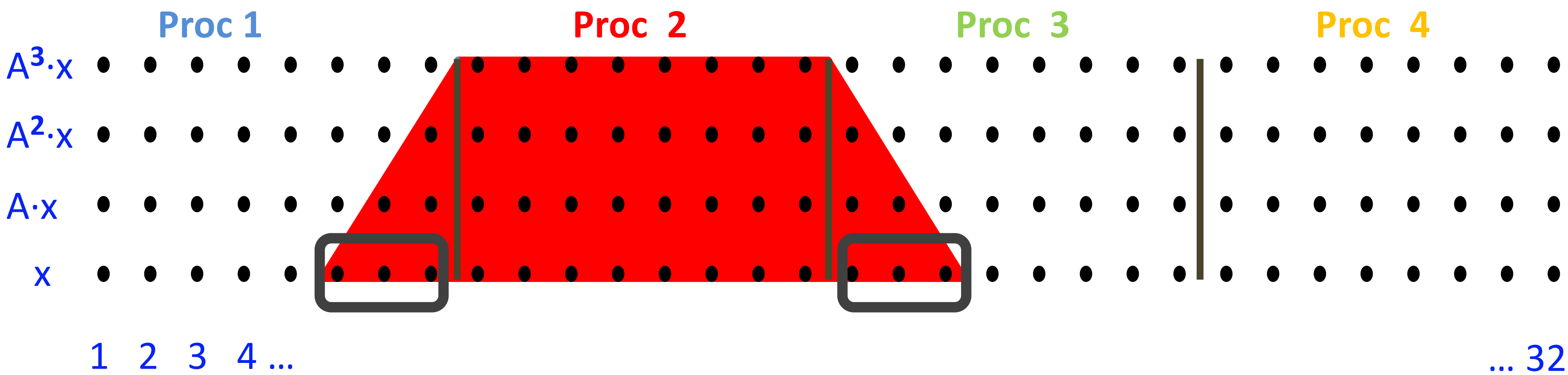
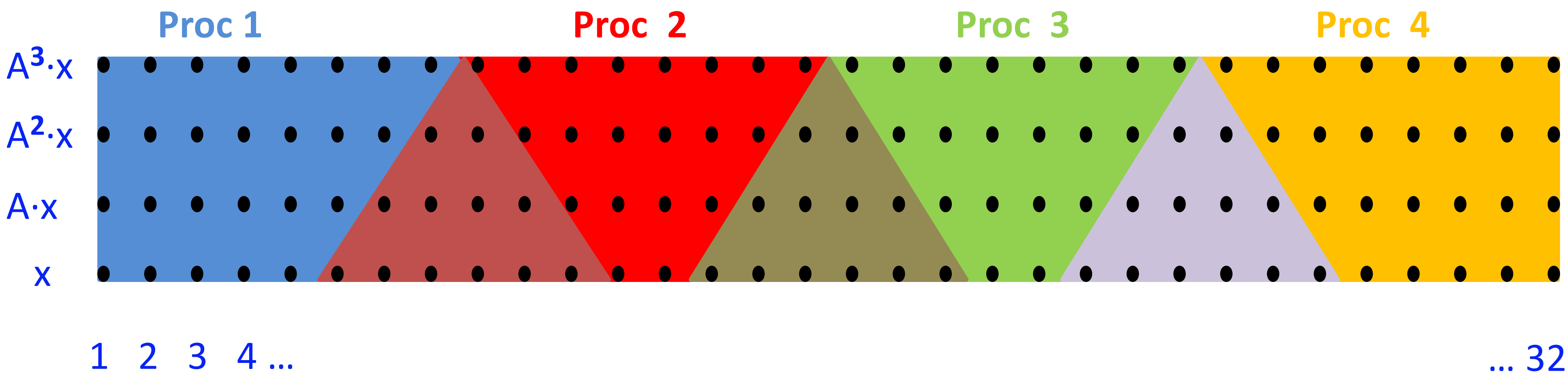
- Parallel Algorithm



- Example: A tridiagonal, n=32, k=3
- Each processor communicates once with neighbors

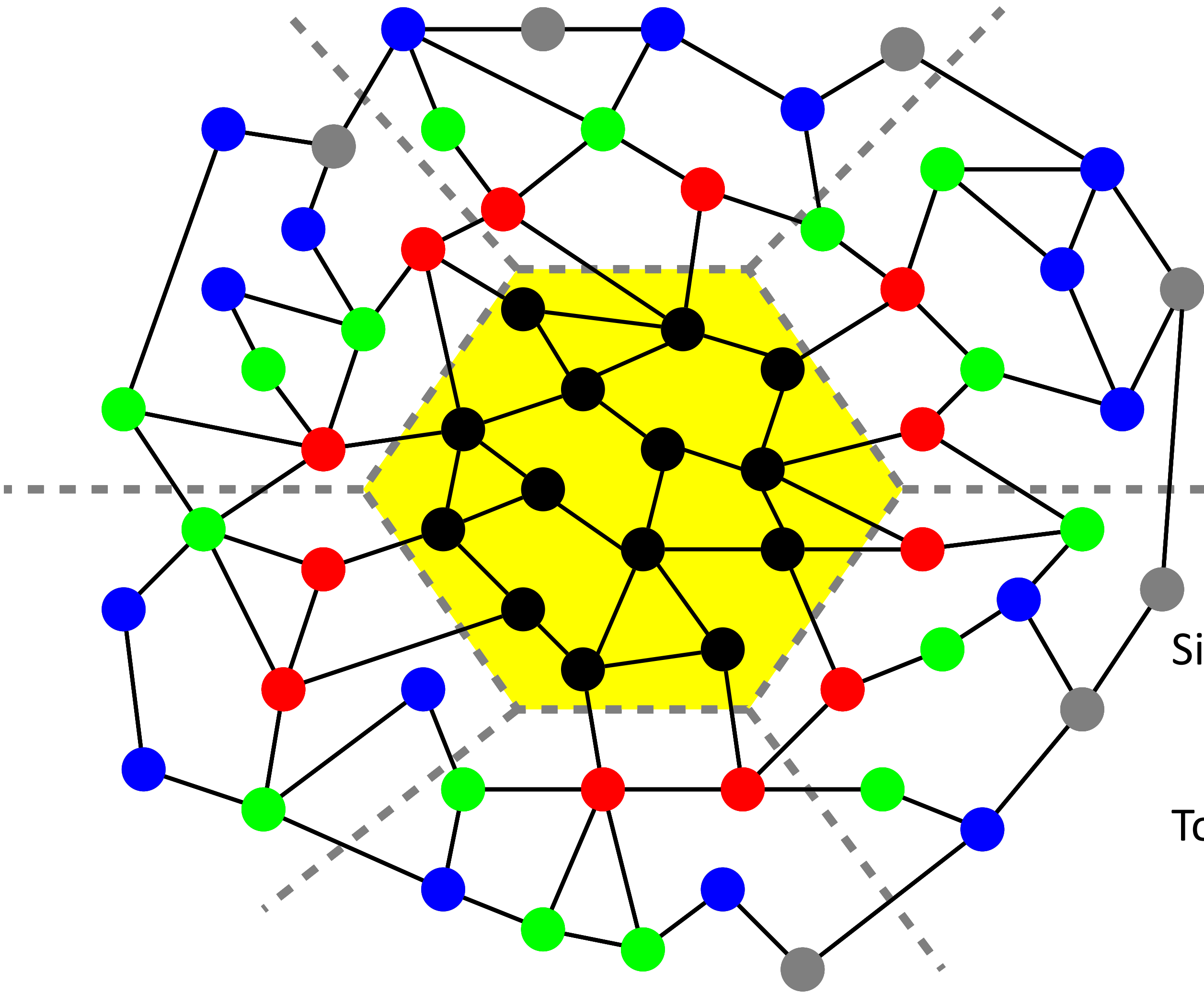
Communication Avoiding Kernels:
The Matrix Powers Kernel : $[Ax, A^2x, \ldots, A^kx]$

- Replace k iterations of $y = A \cdot x$ with $[Ax, A^2x, \ldots, A^kx]$
- Parallel Algorithm



- Example: A tridiagonal, n=32, k=3
- Each processor works on (overlapping) trapezoid

The Matrix Powers Kernel : $[Ax, A^2x, \ldots, A^kx]$ on a general matrix (nearest k neighbors on a graph)

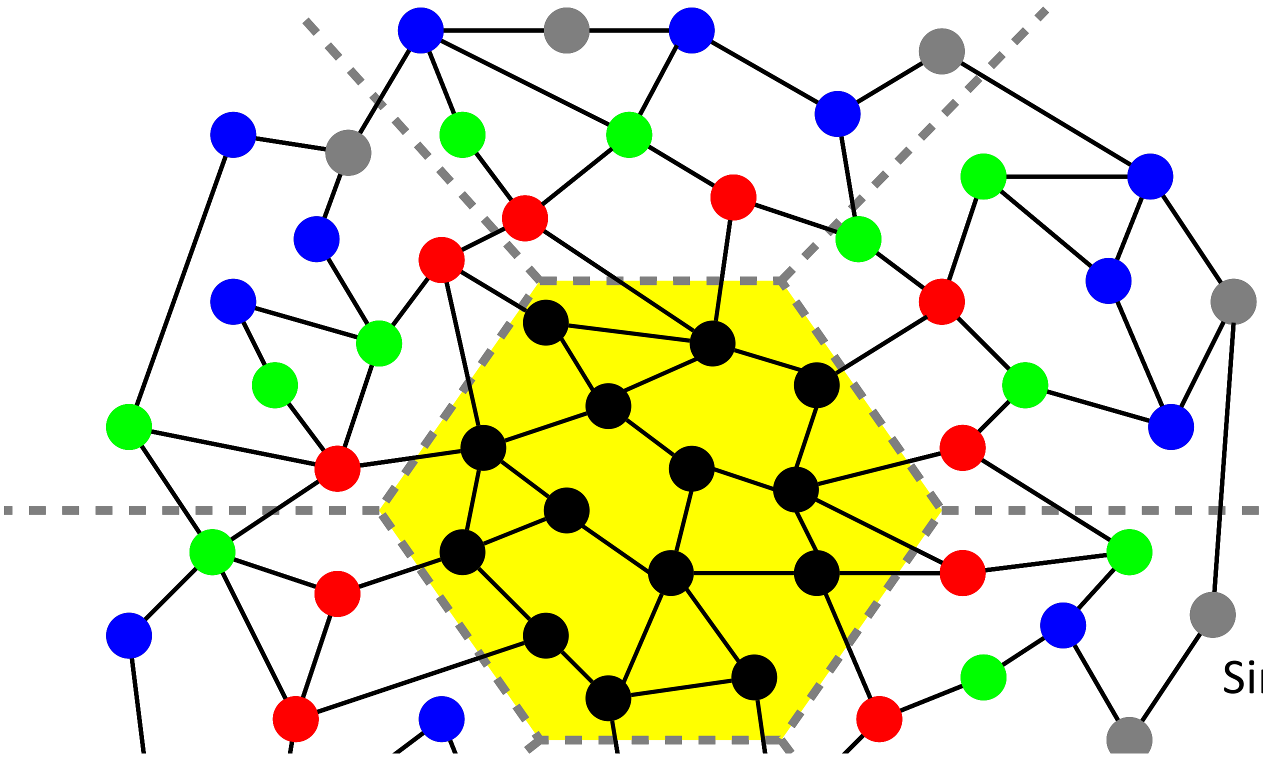


Simple block-row partitioning → (hyper)graph partitioning

Top-to-bottom processing → Traveling Salesman Problem

Same idea for general sparse matrices: k-wide neighboring region

The Matrix Powers Kernel : $[Ax, A^2x, \ldots, A^kx]$ on a general matrix (nearest k neighbors on a graph)



Simple block-row partitioning ➔
(hyper)graph partitioning

Alappat et al, "Level-Based Blocking for Sparse Matrices: Sparse Matrix-Power-Vector Multiplication", 2023
Builds on Alappat et al, SIAG on Supercomputing Best Paper Prize, 2024

CA-BiCGStab

Compute $r_0 = b - Ax_0$. Choose $r_0^\star$ arbitrary.
Set $p_0 = r_0$, $q_{-1} = 0_{N\times 1}$.
For $k = 0, 1, \ldots,$ until convergence, Do

$P = [p_{sk}, Ap_{sk}, \ldots, A^s p_{sk}]$
$Q = [q_{sk-1}, Aq_{sk-1}, \ldots, A^s q_{sk-1}]$
$R = [r_{sk}, Ar_{sk}, \ldots, A^s r_{sk}]$
//Compute the $1 \times (3s+3)$ Gram vector.
$g = (r_0^\star)^T [P, Q, R]$
//Compute the $(3s+3) \times (3s+3)$ Gram matrix

$$G = \begin{bmatrix} P^T \\ Q^T \\ R^T \end{bmatrix} \begin{bmatrix} P & Q & R \end{bmatrix}$$

For $\ell = 0$ to s,

$$b_{sk}^\ell = \left[B_1(:,\ell)^T, 0_{s+1}^T, 0_{s+1}^T\right]^T$$

$$c_{sk-1}^\ell = \left[0_{s+1}^T, B_2(:,\ell)^T, 0_{s+1}^T\right]^T$$

$$d_{sk}^\ell = \left[0_{s+1}^T, 0_{s+1}^T, B_3(:,\ell)^T\right]^T$$

For $j = 0$ to $\left\lfloor \frac{s}{2} \right\rfloor - 1$, Do

$$\alpha_{sk+j} = \frac{<g, d_{sk+j}^0>}{<g, b_{sk+j}^1>}$$

$$q_{sk+j} = r_{sk+j} - \alpha_{sk+j}[P, Q, R] b_{sk+j}^1$$

For $\ell = 0$ to $s - 2j + 1$, Do

$$c_{sk+j}^\ell = d_{sk+j}^\ell - \alpha_{sk+j} b_{sk+j-1}^{\ell+1}$$

//such that $[P, Q, R]\, c_{sk+j}^\ell = A^\ell q_{sk+j}$

$$\omega_{sk+j} = \frac{<c_{sk+j+1}^1, Gc_{sk+j+1}^0>}{<c_{sk+j+1}^1, Gc_{sk+j+1}^1>}$$

$$x_{sk+j+1} = x_{sk+j} + \alpha_{sk+j} p_{sk+j} + \omega_{sk+j} q_{sk+j}$$

$$r_{sk+j+1} = q_{sk+j} - \omega_{sk+j}[P, Q, R] c_{sk+j+1}^1$$

For $\ell = 0$ to $s - 2j$, Do

$$d_{sk+j+1}^\ell = c_{sk+j+1}^\ell - \omega_{sk+j} c_{sk+j+1}^{\ell+1}$$

//such that $[P, Q, R]\, d_{sk+j+1}^\ell = A^\ell r_{sk+j+1}$

$$\beta_{sk+j} = \frac{<g, d_{sk+j+1}^0>}{<g, d_{sk+j}^0>} \times \frac{\alpha}{\omega}$$

$$p_{sk+j+1} = r_{sk+j+1} + \beta_{sk+j} p_{sk+j} - \beta_{sk+j}\omega_{sk+j}[P, Q, R] b_{sk+j}^1$$

For $\ell = 0$ to $s - 2j$, Do

$$b_{sk+j+1}^\ell = d_{sk+j+1}^\ell + \beta_{sk+j} b_{sk+j}^\ell - \beta_{sk+j}\omega_{sk+j} b_{sk+j}^{\ell+1}$$

//such that $[P, Q, R]\, b_{sk+j+1}^\ell = A^\ell p_{sk+j+1}$.

EndDo

EndDo

1. Compute $r_0 := b - Ax_0$; r_0^* arbitrary;
2. $p_0 := r_0$.
3. For $j = 0, 1, \ldots,$ until convergence Do:
4. $\alpha_j := (r_j, r_0^*)/(Ap_j, r_0^*)$
5. $s_j := r_j - \alpha_j Ap_j$
6. $\omega_j := (As_j, s_j)/(As_j, As_j)$
7. $x_{j+1} := x_j + \alpha_j p_j + \omega_j s_j$
8. $r_{j+1} := s_j - \omega_j As_j$
9. $\beta_j := \frac{(r_{j+1}, r_0^*)}{(r_j, r_0^*)} \times \frac{\alpha_j}{\omega_j}$
10. $p_{j+1} := r_{j+1} + \beta_j(p_j - \omega_j Ap_j)$
11. EndDo

Outline

- Linear Algebra
  - Communication Lower Bounds for classical direct linear algebra
  - CA 2.5D Matmul
  - TSQR - Tall-Skinny QR
  - Iterative Methods for linear algebra
- **Machine Learning**
  - **Training Neural Nets – "ImageNet training in minutes"**
  - Convolutional Neural Nets
- And Beyond
  - Extending communication lower bounds and optimal algorithms to general loop nests

Training Neural Nets by Mini-Batch Stochastic Gradient Descent (SGD)

(You, Zhang, Hsieh, D., Keutzer, IPDPS 18)

- Iterate:
  - Pick a mini-batch of B data points
  - Update weights $W = W - \eta \cdot \nabla L(W)$
    - η = learning rate
    - $\nabla L(W)$ = gradient
- Data parallel version on P processors
  - Data partitioned, each processor gets B/P points
  - W_i replicated
  - Each processor computes $\nabla L(W)_i$ wrt its data
  - All-reduce: each processor computes
    $W_i = W_i - (\eta/P) \cdot \Sigma_{i=1}^{P} \nabla L(W)_i$

SGD: $W_i = W_i - (\eta/P) \cdot \sum_{i=1}^{P} \nabla L(W)_i$

- Increase P to go faster: What are the bottlenecks?
- B/P decreases ⇒ less work per processor
  - Small matrix operations ⇒ locally communication bound
- Cost of each reduction $\Sigma_i \nabla L(W)_i$ grows
- Solution: increase B along with P
  - Maintain B/P ⇒ maintain processor efficiency
  - Try to converge in same #epochs (passes over data)
    - Same overall work, fewer reductions
- Oops: Convergence can be much worse
  - Convergence rate, test accuracy

Improving SGD convergence as B grows

- Facebook's strategy: adjust learning rate η
  - Increase B to kB $\Rightarrow$ increase η to $\mathrm{k}\eta$
  - Warmup rule: Start with smaller η, then increase
- Only worked up to B=1K for AlexNet (tried lots of tuning)
- Fix: Add Layer-wise Adaptive Rate Scaling (LARS)
  - $\| W \| / \| \nabla L(W) \|$ can vary by 233x between AlexNet layers
  - Let η be proportional to $\| W \| / \| \nabla L(W) \|$
  - (You, Gitman, Ginsburg, 2017)
  - Also need momentum, weight decay

ImageNet Training in Minutes

Speedup for AlexNet (for batchsize = 32K, changed LRN to BN)

Batch Size	Epochs	Top-1 Accuracy	Platform	Time
256	100	58.7%	8-core + K20 GPU	144 hrs
512	100	58.8%	DGX-1 station	6h 10m
4096	100	58.4%	DGX-1 station	2h 19m
32k	100	58.6%	512 KNLs	24m
32k	100	58.6%	1024 CPUs	**11m**

Speedup for ResNet50

Batch Size	Epochs	Top-1 Accuracy	Platform	Time
32	90	75.3%	CPU + M40 GPU	336h
256	90	75.3%	16 KNLs	45h
32K	90	75.4%	512 KNLs	60m
32K	90	75.4%	1600 CPUs	32m
32K	90	75.4%	2048 KNLs	**20m**

135x (45h → 20m)

ImageNet Training in Minutes

- Best Paper Prize at ICPP 2018
- Open Source in Caffe, NVIDIA Caffe, Facebook Caffe 2 (PyTorch)
- Media coverage by CACM, EureKalert, Intel, NSF, Science Daily, Science NewsLine, etc.
- Subsequent work at Tencent reached 4 minutes
- LARS adopted by industry standard benchmark MLPerf in 2019

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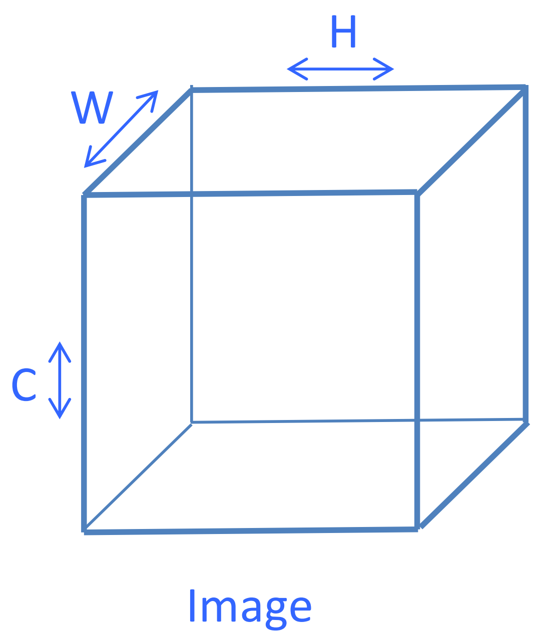
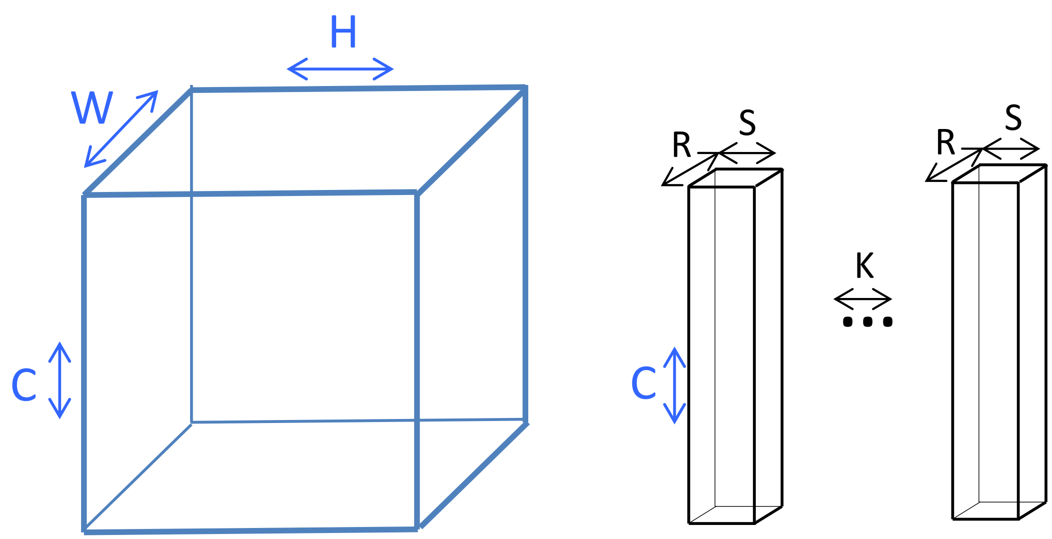
What CNNs compute



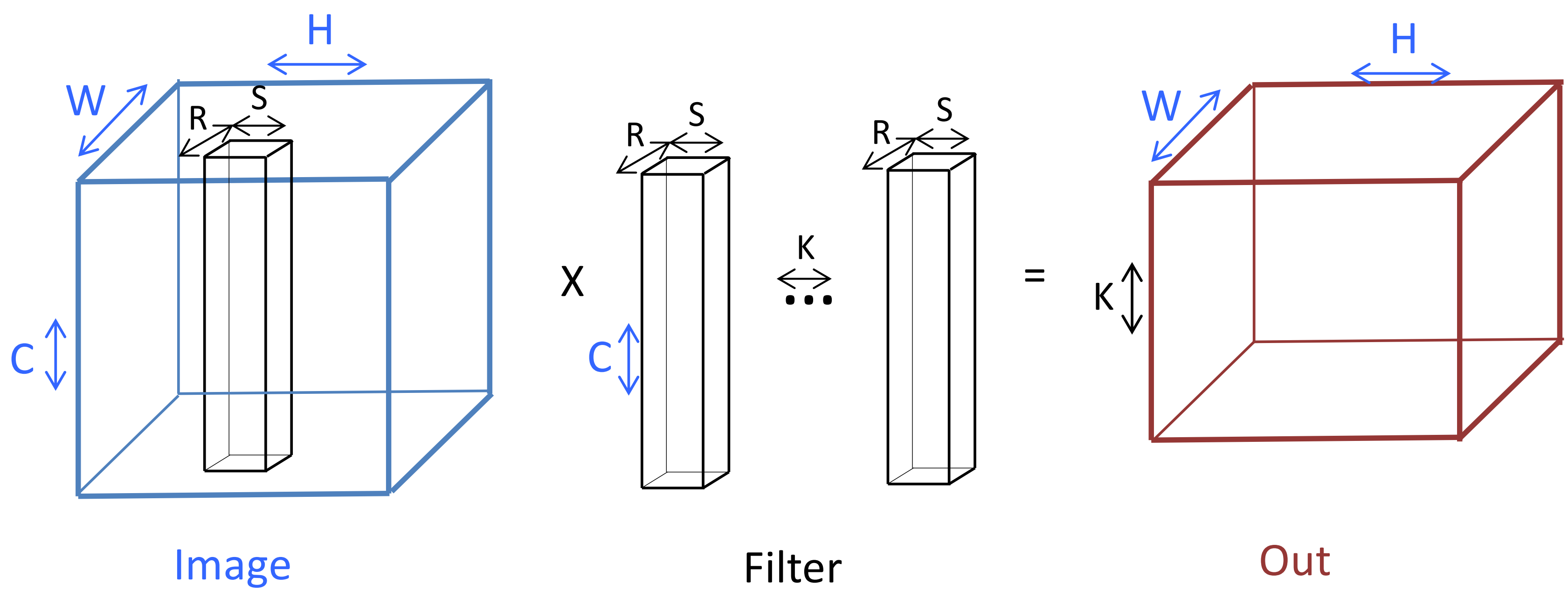
What CNNs compute



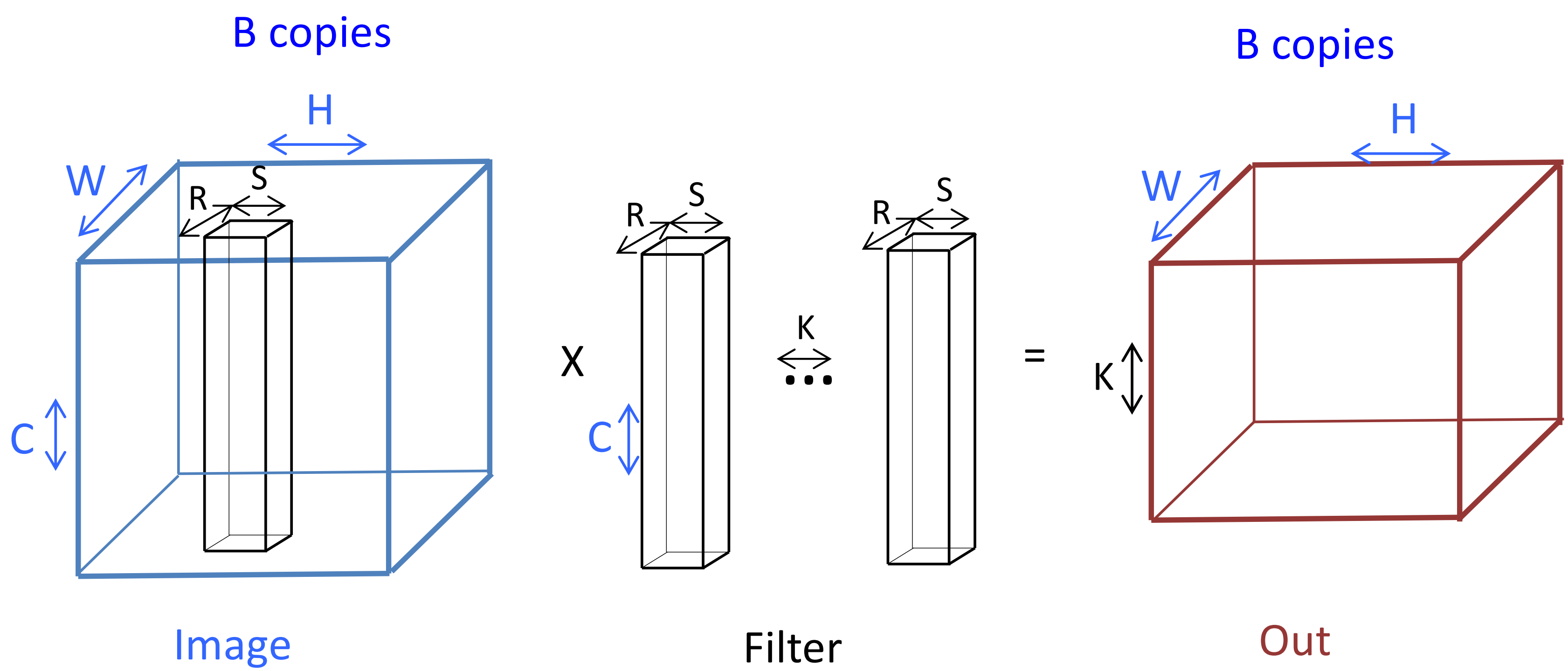
Image

Filter

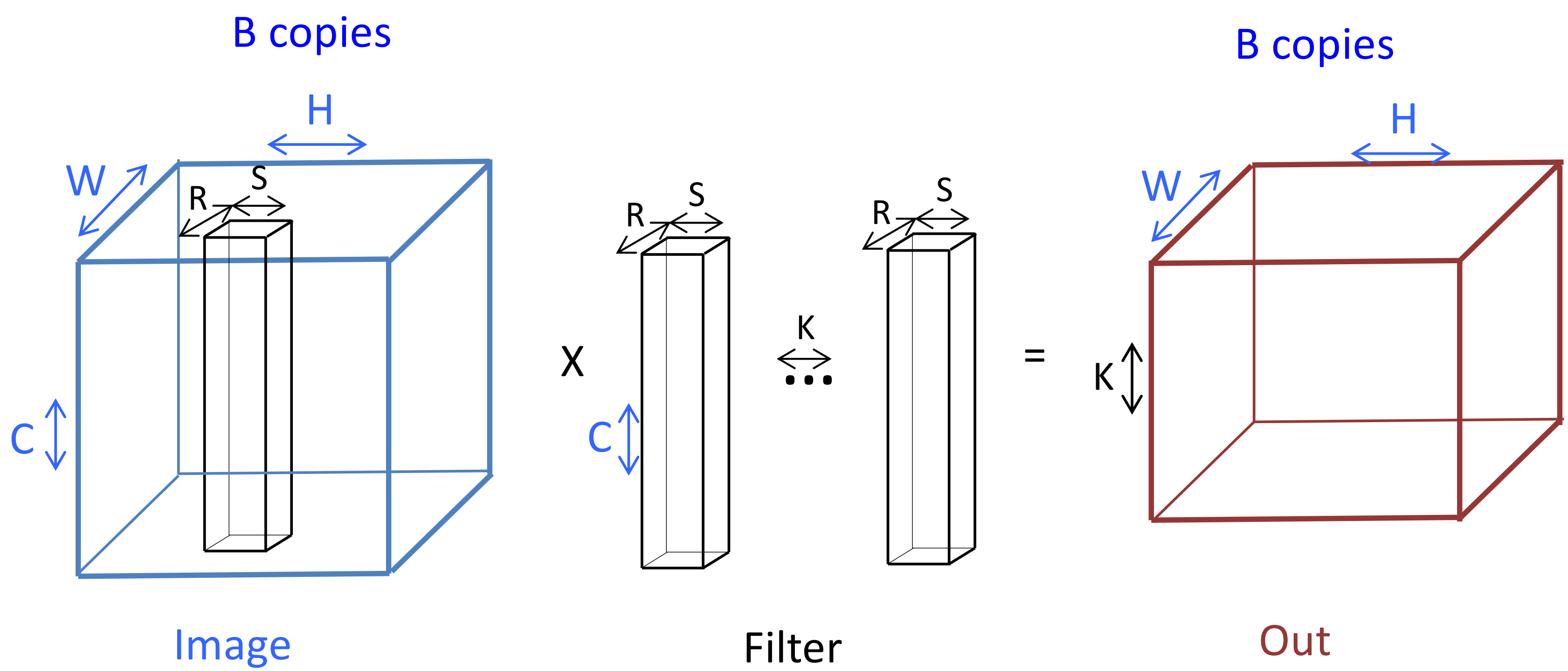
What CNNs compute



What CNNs compute

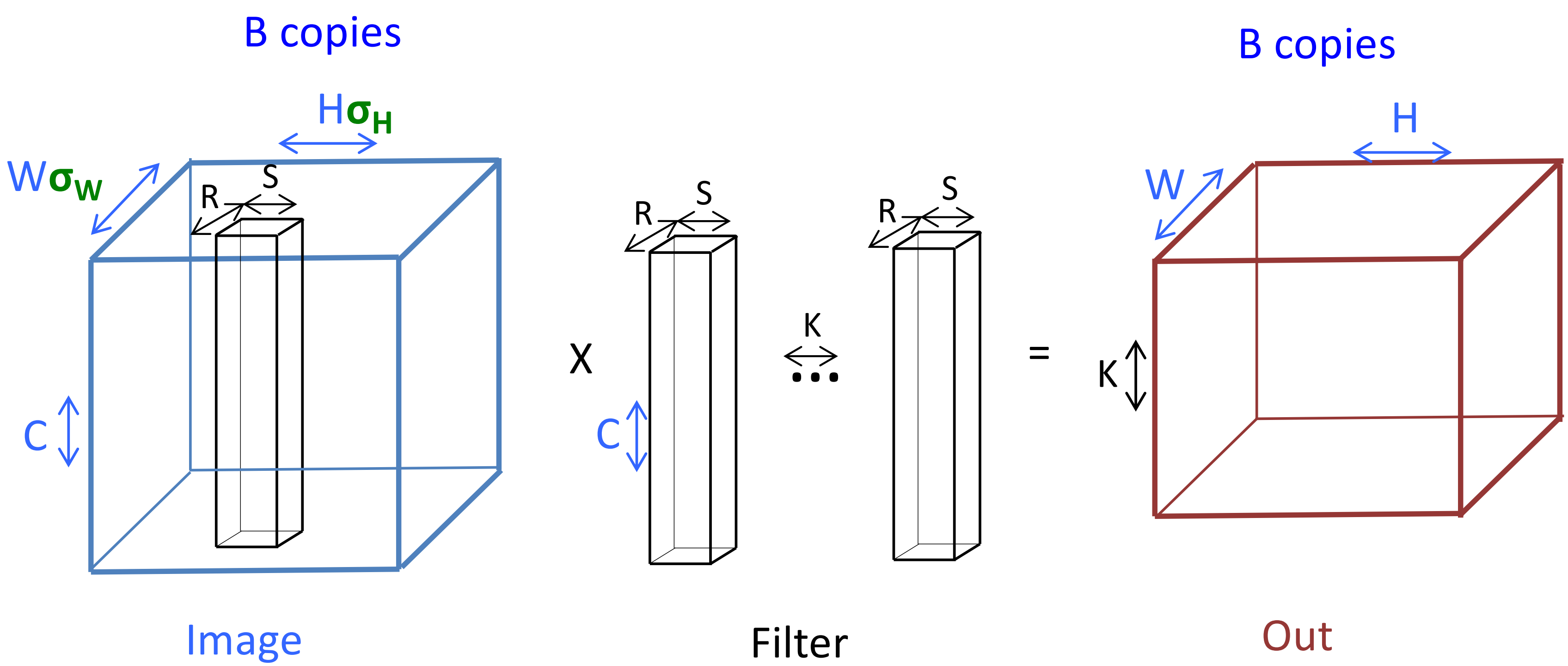


What CNNs compute



for k=1:K,   for h=1:H,   for w=1:W,   for r=1:R,
for s=1:S,   for c=1:C,   for b=1:B
    Out(k, h, w, b) += Image(r+w, s+h, c, b) * Filter(k, r, s, c)

What CNNs compute



for k=1:K, for h=1:H, for w=1:W, for r=1:R,
for s=1:S, for c=1:C, for b=1:B

$$\text{Out}(k, h, w, b) \mathrel{+}= \text{Image}(r+\sigma_W w, s+\sigma_H h, c, b) * \text{Filter}(k, r, s, c)$$

Communication Lower Bound for CNNs

- Let N = #iterations = KHWRSCB, M = cache size
- #words moved = Ω(max(... 5 terms

  BKHW, ... size of Out

  $\sigma_H\sigma_W$BCWH, ... size of Image

  CKRS, ... size of Filter

  N/M, ... lower bound for large loop bounds

  $N/(M^{1/2}(RS/(\sigma_H\sigma_W))^{1/2})$ ... lower bound for small filters)
- Any one of 5 terms may be largest
- Bottommost bound beats matmul by factor $(RS/(\sigma_H\sigma_W))^{1/2}$
  - Applies in common case when data does not fit in cache, but one RxS filter does
  - Tile needed to attain N/M too big to fit in loop bounds
- Thm: Always attainable! (computer generated proof)
  - Beats im2col in data movement for various practical sizes
- Improved constants in PASC‘22
- Chen/Han/Wang (arxiv:1911.05662v3): HW accelerator

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Communication lower bounds and optimal algorithms for general loop nests

- for i = 1:n, for j=1:n, for k = 1:n

  C(i,j) = C(i,j) + A(i,k)*B(k,j)
- #Words moved between main memory and cache of size M = $\Omega(n^3 / M^{1/2})$, attainable
- For $(i_1,i_2,\ldots i_k) \in S \subseteq \mathbb{Z}^k$, do something with
  - $A(i_1)$, $B(i_2, i_3+i_4)$, $C(i_1-i_2, i_2+3*i_3- 5*i_4 +2, \ldots)$, …
- Thm: #Words moved = $\Omega(|S| / M^{e_{HBL}})$
  (Christ, D., Knight, Scanlon, Yelick)
  - HBL = Hölder / Brascamp / Lieb
  - Uses results by Christ, Tao, others
- Thm: There exists an optimal tiling that attains this lower bound (D., Rusciano)

What’s left?

- Dealing with small loop bounds
  - Ex: Matvec special case of Matmul, not optimizable
  - Special cases: CNNs
  - Thm: If all subscripts like (i),(i,j), etc, and S = parallelepiped, ∃ tighter, attainable lower bound (D., Dinh)
- Dealing with dependencies
  - Special cases: Linear algebra outside matmul, Floyd-Warshall, … open problems!
- More realistic performance models than α, β, γ
  - Variable precision
  - Heterogeneous processors, accelerators, network topologies, differing costs of read and writes, …
- Sketching – can beat matmul
- Need to automate! (i.e. compilers)
  - https://iocomplexity.corse.inria.fr/iolb

Collaborators and Supporters

- **James Demmel**, **Kathy Yelick**, Vivek Bharadwaj, Grace Dinh, Tianyu Liang
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- bebop.cs.berkeley.edu

For more details

- Bebop.cs.berkeley.edu
  - 155 page linear algebra survey in Acta Numerica (2014)
  - Book in progress (with Ballard, Carson, Grigori)
- CS267 – Berkeley's Parallel Computing Course
  - Slides available at https://sites.google.com/lbl.gov/cs267-spr2025
  - Earlier recorded lectures also available at https://sites.google.com/lbl.gov/cs267-spr2022

Summary

Time to redesign all

linear algebra, machine learning, n-body, …
algorithms and software, and compilers

Don't Communic…

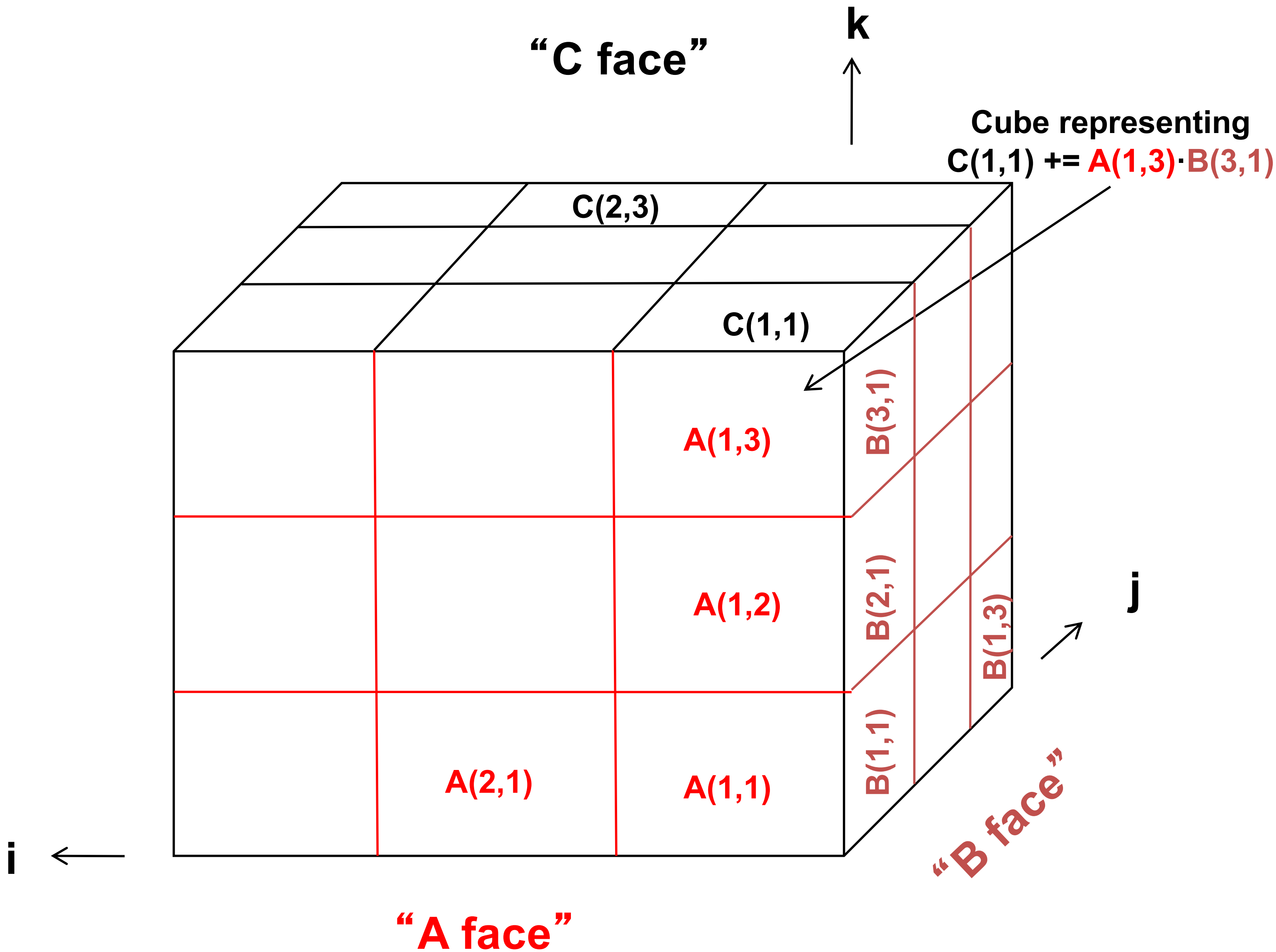
77

Backup slides

Proof of Communication Lower Bound on C = A·B (1/4)

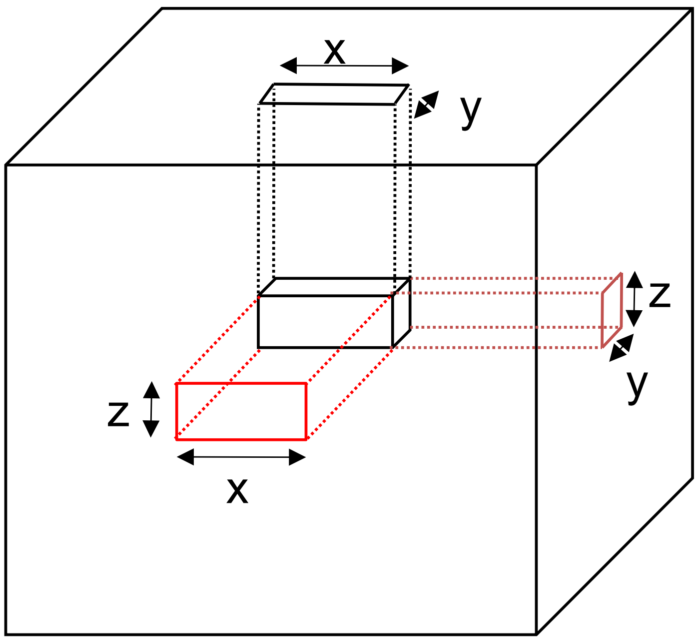
- Basic Counting Argument:
  - Only M entries of A, B and C are available in cache
  - Find an upper bound F on the number of different iterations C(i,j) = C(i,j) + A(i,k)*B(k,j) we can perform
  - Need to refill cache n^3/F times to complete algorithm
  - Need to read/write at least $M\ n^3/\ F$ words to/from cache
- Represent iterations and data geometrically

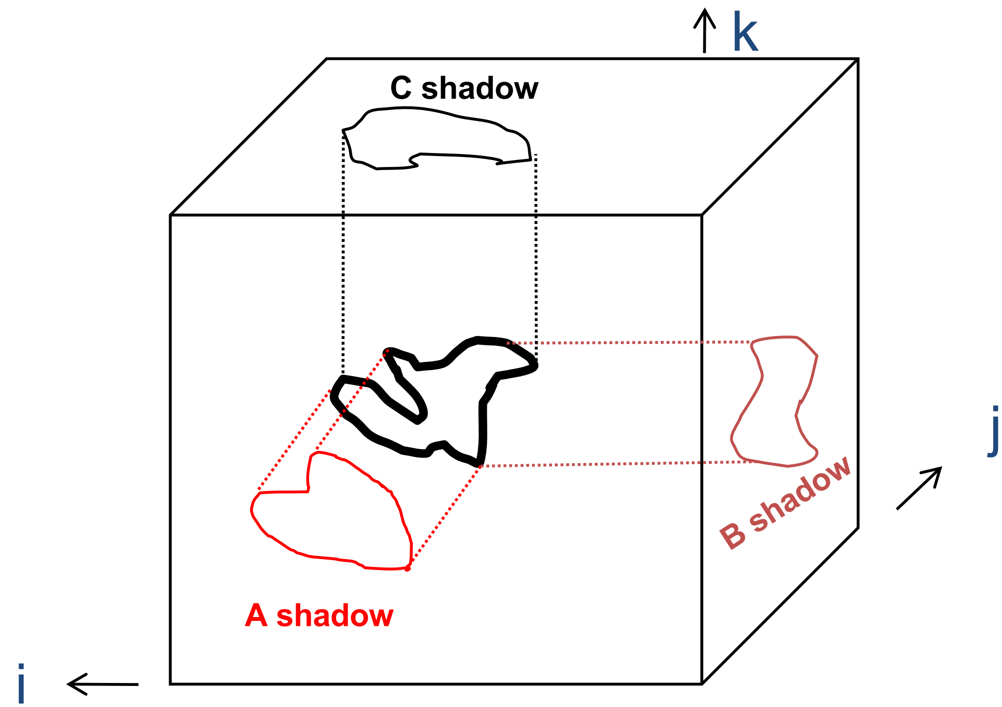
Proof of Communication Lower Bound on C = A·B (2/4)



- If we have at most M "A squares", "B squares", and "C squares" on faces, how many cubes can we have?

Proof of Communication Lower Bound on C = A·B (3/4)





(i,k) is in **A shadow** if (i,j,k) in 3D set
(j,k) is in **B shadow** if (i,j,k) in 3D set
(i,j) is in C shadow if (i,j,k) in 3D set

 # cubes in black box with side lengths x, y and z
= Volume of black box
= x·y·z
= (xz · zy · yx)$^{1/2}$
= (#A□s · #B□s · #C□s)$^{1/2}$

Thm (Loomis & Whitney, 1949)
 # cubes in 3D set = Volume of 3D set
≤ (area(A shadow) · area(B shadow) · area(C shadow)) $^{1/2}$

Proof of Communication Lower Bound on $C = A \cdot B$ (4/4)

- # loop iterations doable with M words of data = #cubes

  $\leq$ (area(A shadow) · area(B shadow) ·area(C shadow)) $^{1/2}$

  $\leq (M \cdot M \cdot M)^{1/2} = M^{3/2} = F$
- Need to read/write at least $M n^3 / F = \Omega(n^3/M^{1/2}) = \Omega(\#\text{loop iterations} / M^{1/2})$ words to/from cache